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On ghost fermions

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Abstract. The path integral for ghost fermions, which is heuristically made use of in the Batalin-Fradkin-Vilkovisky approach to quantization of constrained systems, is derived from first principles. The derivation turns out to be rather different from that of physical fermions since the definition of Dirac states for ghost fermions is subtle. With these results at hand, it is then shown that the nonminimal extension of the Becchi-Rouet-Stora-Tyutin operator must be chosen differently from the notorious choice made in the literature in order to avoid the boundary terms that have always plagued earlier treatments. Furthermore it is pointed out that the elimination of states with nonzero ghost number requires the introduction of a thermodynamic potential for ghosts; the reason is that Schwarz's Lefschetz formula for the partition function of the time-evolution operator is not capable, despite claims to the contrary, to get rid of nonzero ghost number states on its own. Finally, we comment on the problems of global topological nature that one faces in the attempt to obtain the solutions of the Dirac condition for physical states in a configuration space of nontrivial geometry; such complications give rise to anomalies that do not obey the Wess-Zumino consistency conditions.

Introduction and summary

Systems with first class constraints, of which abelian and nonabelian gauge theories are prime examples, are rather perfectly understood classically through Marsden-Weinstein reduction [1]. The quantization of such systems is achieved by means of the Batalin-Fradkin-Vilkovisky (BFV) approach [2–4], being based on the Becchi-Rouet-Stora-Tyutin (BRST) construction which in turn follows from Faddeev's formula [5] as the essential ingredient; hence, just the opposite strategy is pursued since, instead of restricting the phase space, it is enlarged by introducing additional ghost degrees of freedom.

But there are still some open problems, such as the construction of physical states with finite norm in the operator approach (see, e.g., [6,7]). What is also missing is a proper understanding of the path integral for ghost fermions, which is only formally written down in the BFV approach, without clarifying its origin. Furthermore, it is not known in which way the partition function of the BFV system by itself manages that the cohomology collapses at zero ghost number [8]. It is the purpose of the present paper to contribute to a solution of these problems.

In particular, we attempt to give a derivation of the path integral for ghost fermions at a comparable level of rigor as that invested for the other ones (see, e.g., [9]). Let us recall, there are three different types of path integrals which were investigated in the past and are rather well understood by now. These are

– the Feynman path integral in its original lagrangian version; its hamiltonian version [10,12,13] is, to cite Henneaux ([11], p.65), "full of subtleties", but it is actually the one being required in the context of constrained systems.
– the coherent state path integral, which in many aspects is simpler than the Feynman type of path integral; it has the virtue to admit a rather 'coherent' treatment of bosons and Dirac fermions [14,15], but is not applicable to ghost fermions.
– the Berezin path integral [16], which uses the symplectic structure of the phase space for fermions and bosons through the Weyl approach to quantization [17,18].

What has been brought forward [19] as a possible candidate for a path integral of ghost fermions, is Berezin's variant. This, however, leads to boundary conditions, being essentially different from that obtained for a Feynman path integral in hamiltonian form; but it is the latter type of boundary conditions that is needed here. Hence, standard techniques fail for the case at hand.

We approach the problem by extending earlier work of Marnelius [20]. For this, we first give a construction of the Dirac basis for ghost fermions, which is not at all straightforward since it requires the introduction of an unconventional kind of (real) coherent states. Then we are able to derive the corresponding path integral by following rather standard lines so that the trace and supertrace of a zero ghost number operator can be expressed as a functional integral. These matters form the content of the first section, which are applied in the second section to the BFV system. There it is shown that the nonminimal extension of the BRST operator requires modification in order to achieve that the BFV partition function really is invariant under BRST transformation, whereas the notorious choice made in the literature [2,19] suffers from boundary terms [11, 19,21] that destroy its invariance. This modification also admits to give a proof of the Fradkin-Vilkovisky [2] theorem that rectifies some weak points of the original version [3]. We then turn to the operator treatment of the BRST approach, where it is known from Schwarz's work [22] that it is the supertrace in the ghost sector, which must be used in order to achieve that only the contributions from the cohomology groups survive in the partition function. Using the results of the first section, we can then express the Lefschetz formula as a functional integral which, however, involves the contributions of all cohomologies and not, as one wants, of the zero cohomology only. This problem has also been seen and dealt with recently in [8], but our answer is different. As we believe, the problem can only be settled by introducing a thermodynamic potential for the ghosts. Then one can get rid of the nonzero cohomologies by isolating that part of the total partition function, which is independent of the thermodynamic potential. In the concluding section it is demonstrated on the example of abelian Chern-Simons theory in the plane and on the torus that, through the integrated version of the Dirac condition for physical states, an anomaly [23] is encountered since the associated group two-cocycle is generally non trivial.

1 Ghost fermions

We want to model the fermionic analogue of bosonic momentum operators $\hat{p}_i$ and generalized coordinate operators $\hat{q}^j$, denoted by $\hat{\zeta}_a$ and $\hat{\eta}^b$ in the following. The treatment of these matters in the literature, if given at all, is both controversial and incomplete. For example, Berezin and Marinov [17] remark that "in the Grassmann phase space one cannot use the coordinate-momentum language, and it is impossible to define analogue of the Feynman path integral in the coordinate (or momentum) space." This statement is indeed true for physical real fermions. As we will show, however, the above verdict may be overcome for ghost fermions.

What will turn out to be a nontrivial affair is to construct a Dirac basis for such unphysical fermions. Recall in this context that a proper definition of (bosonic) Dirac kets, which are needed for the Feynman path integral approach to quantization, is a subtle issue that requires the concept of Gel'fand triplets [24,25]. Hence it should come to no surprise that also the construction of Dirac states for ghost fermions will involve some subtleties.

1.1 Schrödinger representation

It is natural to assume that the operators, corresponding to the real fermionic momentum variables ζ_a and coordinate variables η^b, must obey the anticommutation relations

$$[\hat{\zeta}_a, \hat{\eta}^b]_+ = \delta_a{}^b \tag{1.1}$$

where $a, b \in \{1, \ldots, m\}$. The operators $\hat{\zeta}_a$ and $\hat{\eta}^b$ are supposed to be selfadjoint, in a sense to be made precise, because the corresponding Grassmann variables are real by assumption. It is for this reason, that the factor $\sqrt{-1}$ is missing on the right-hand side of the basic anticommutator.

A straightforward strategy to find a realization of the algebra of operators with the above defining relations is to proceed along the lines of the bosonic case. So we introduce fermionic 'Schrödinger' wave functions (cf. also [26, 27])

$$\psi(\eta) = \sum_{p=0}^{m} \frac{1}{p!} \eta^{a_1} \cdots \eta^{a_p} \, \psi_{a_1 \cdots a_p} \tag{1.2}$$

of the Grassmannian configuration space variables η^a; they are real in the sense $(\eta^a)^* = \eta^a$, with the $*$-involution inverting the order of the factors: $(\eta^{a_1} \cdots \eta^{a_p})^* = \eta^{a_p} \cdots \eta^{a_1}$. The completely antisymmetric coefficients $\psi_{a_1 \cdots a_p}$ are assumed to take complex values. On such wave functions, the operators $\hat{\zeta}_a$ and $\hat{\eta}^b$ are defined to act as

$$\hat{\eta}^b \psi(\eta) = \eta^b \psi(\eta) \qquad \hat{\zeta}_a \psi(\eta) = \frac{\partial}{\partial \eta^a} \psi(\eta). \tag{1.3}$$

Here, the derivative must necessarily act from the left in order to reproduce the fundamental anticommutator.

We now turn to the definition of selfadjointness for the above operator realization. So a sesquilinear form $\langle \psi | \psi' \rangle$ on these Grassmann valued wave functions must be introduced; a natural choice is

$$\begin{aligned} \langle \psi | \psi' \rangle &= \int d^m \eta \, \psi(\eta)^* \psi'(\eta) \\ &= \langle \psi | \psi' \rangle \\ &= \sum_p \frac{(-1)^{\binom{p}{2}}}{(m-p)! p!} \varepsilon^{a_1 \cdots a_p a_{p+1} \cdots a_m} \psi^*_{a_1 \cdots a_p} \psi'_{a_{p+1} \cdots a_m}. \end{aligned} \tag{1.4}$$

With respect to this inner product the operators $\hat{\zeta}_a$ and $\hat{\eta}^b$ are selfadjoint; on using the rules of Grassmann calculus, the proof is by direct verification. There is also a ghost number operator [28] available

$$\hat{N} = \frac{1}{2} \left(\hat{\eta}^a \hat{\zeta}_a - \hat{\zeta}_a \hat{\eta}^a \right) \tag{1.5}$$

being constructed such that it is skew adjoint with respect to the inner product; it counts the momentum operators as $+1$ and the coordinate operators as -1. On the subspace of functions $\psi_p(\eta) = \frac{1}{p!} \eta^{a_1} \cdots \eta^{a_p} \, \psi_{a_1 \cdots a_p}$ of naïve Grassmann degree p the operator $\hat{N}$ is diagonal with eigenvalue $p - \frac{m}{2}$, which is $\binom{m}{p}$-fold degenerate.

For the investigation of the properties of this sesquilinear form, the above explicit expression (1.4) is not very useful. Instead, it is advantageous to turn to a (equivalent) Fock type of representation for the wave function

$$\psi(\eta) = \sum_{n_1, \ldots, n_m = 0}^{1} (\eta^1)^{n_1} \cdots (\eta^m)^{n_m} \psi_{n_1 \cdots n_m}. \tag{1.6}$$

We then obtain the alternative expression

$$\langle\psi|\psi'\rangle = \sum_{n_1\cdots n_m} (-1)^{\sum_i n_i(i-1)} \psi^*_{n_1\cdots n_m} \psi'_{\bar n_1\cdots \bar n_m} \qquad (1.7)$$

where $\bar n_i = 1-n_i$, from which one infers that the sesquilinear form is nondegenerate.

However, this nondegenerate sesquilinear form is generally not hermitian (or symmetric for real functions) because

$$\langle\psi|\psi'\rangle^* = (-1)^{\binom{m}{2}}\langle\psi'|\psi\rangle. \qquad (1.8)$$

We could enforce hermiticity by a simple redefinition of the inner product; but in the field theoretic case (as well as for functions with real coefficients) this approach is not amenable since this would yield an unwanted accumulation of phase factors. So we must take both the indefiniteness and the non-hermiticity at face.

The indefiniteness of the inner product reflects the properties of the corresponding Clifford algebra with generating elements $\hat\xi_\alpha$, given by

$$\hat\xi_a = \hat\zeta_a \qquad \hat\xi_{m+a} = \hat\eta_a \qquad (1.9)$$

where $\alpha = 1,\ldots,2m$. The defining relations are $\hat\xi_\alpha\hat\xi_\beta + \hat\xi_\beta\hat\xi_\alpha = g_{\alpha\beta}$, with the metric tensor

$$g = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} \qquad (1.10)$$

indeed being indefinite.

This property of unphysical real fermions is in marked contrast to the properties of complex physical fermions. For these, the fundamental anticommutator is

$$[\hat\psi^*_A(\mathbf{x}), \hat\psi^B(\mathbf{y})]_+ = \delta_A{}^B\delta(\mathbf{x},\mathbf{y}). \qquad (1.11)$$

Omitting the x-dependence and spinor indices A, B altogether, the corresponding (real) Clifford algebra generators are

$$\hat\xi_1 = \frac{1}{\sqrt2}(\hat\psi + \hat\psi^*) \qquad \hat\xi_2 = \frac{i}{\sqrt2}(\hat\psi - \hat\psi^*) \qquad (1.12)$$

which yield

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (1.13)$$

that is, a positive definite metric. For these physical real fermions, however, there is no natural splitting of ξ_1 and ξ_2 into a coordinate and momentum; the choice of a real polarization would destroy rotational invariance. Only the holomorphic polarization is available, which is made use of in the standard coherent state representation [15].

On the other hand, for real unphysical fermions, one could try to pass in analogy to (1.12) to operators $a = (\zeta - i\eta)/\sqrt2$ and $a^* = (\zeta + i\eta)/\sqrt2$ obeying $(a^*)^* = a$; however, since $[a, a^*]_+ = 0$ they cannot be interpreted as fermionic creation and annihilation operators. Hence, for ghost fermions a complex structure does not make sense.

1.2 Vector space realization

We want to give a conventional matrix realization of the operators $\hat\zeta_a$ and $\hat\eta^b$ on a 2^m-dimensional complex vector space, i.e. without taking recourse to Grassmann variables. This construction will be needed in the following subsection.

For this purpose, we choose a complex linear space of dimension 2^m with basis $|n_1,\ldots,n_m\rangle$ where $n_a = 0, 1$, the general element of which we write in the form

$$|\psi\rangle = \sum_{n_1\cdots n_m} |n_1,\ldots,n_m\rangle\psi(n_1,\ldots,n_m). \qquad (1.14)$$

Taking (1.3) as a guiding principle, the action of the momentum and coordinate operators on the basis is defined to be

$$\begin{aligned} &\hat\zeta_a|n_1,\ldots,n_a,\ldots,n_m\rangle \\ &\quad = (-1)^{n_1+\cdots+n_{a-1}}\bar n_a|n_1,\ldots,\bar n_a,\ldots,n_m\rangle \\ &\hat\eta_a|n_1,\ldots,n_a,\ldots,n_m\rangle \\ &\quad = (-1)^{n_1+\cdots+n_{a-1}} n_a|n_1,\ldots,\bar n_a,\ldots,n_N\rangle. \end{aligned} \qquad (1.15)$$

Let us introduce the special state $|0\rangle = |0,\ldots,0\rangle$, satisfying $\hat\eta^a|0\rangle = 0$ for all $a \in \{1,\ldots,m\}$, and with the help of which we can generate the whole basis according to

$$(\hat\zeta_1)^{n_1}\cdots(\hat\zeta_m)^{n_m}|0\rangle = |n_1,\ldots,n_m\rangle. \qquad (1.16)$$

Of course, the (1.15) realize nothing else but the standard Fock space construction of the creation operators $\hat\zeta_a$ and the destruction operators $\hat\eta^a$; what is crucially different, however, this is the inner product on the Fock space. In the present case it must be chosen such that the creation and annihilation operators are selfadjoint, whereas in the standard case they are adjoint to one another. For this purpose we imitate (1.7) and define the (nonstandard) inner product to be

$$\begin{aligned} &\langle n_m,\ldots,n_1|n'_1,\ldots,n'_m\rangle \\ &\quad = (-1)^{\sum_a n_a(a-1)}\delta_{\bar n_1 n'_1}\cdots\delta_{\bar n_m n'_m}. \end{aligned} \qquad (1.17)$$

Again, this is nondegenerate, but neither hermitian nor positive definite; in particular, all basis vectors have norm zero. This is the explicit realization of the vector space together with its indefinite inner product, being implicitly encountered in the operator approach to the BRST algebra [28,29]. Below we shall have need of the particular basis vector

$$|\bar 0\rangle = (-1)^{\binom{m}{2}}|1,\ldots,1\rangle \qquad (1.18)$$

which is of 'highest weight' with respect to the destruction operators $\hat\eta^a$ and normalized such that $\langle\bar 0|0\rangle = 1$ holds. What remains to prove is that the matrix representation of the momentum and coordinate operators (1.15) is selfadjoint with respect to (1.17); the proof is by direct verification. The last point to be discussed is the completeness relation. For this purpose, we use

$$\langle n_m,\ldots,n_1|\psi\rangle = (-1)^{\sum_a n_a(a-1)}\psi(\bar n_1,\cdots,\bar n_m) \quad (1.19)$$

and this in turn gives

$$\sum_{n_1\cdots n_m}(-1)^{\sum_a \bar{n}_a(a-1)}|n_1,\ldots,n_m\rangle\langle\bar{n}_m,\ldots,\bar{n}_1| = 1_{2^m} \tag{1.20}$$

which is the result sought for.

It is quite remarkable that along these lines one can circumvent the general representation theory of Clifford algebras on vector spaces of dimension $2m$ with an inner product of zero signature. As is known, the general approach (see [30]) makes essential use of the representation theory of finite groups to prove the existence of a (unique) representation of dimension 2^m, which we here have constructed explicitly.

1.3 Dirac states and their duals

Ultimately, what we want is a path integral for these unphysical real fermions, which must be derived from first principles; but for this it is mandatory to have available a Dirac basis. From the treatment of standard (complex) coherent states (cf. also [20]), we are acquainted with the definition

$$|\eta\rangle = \exp(\hat{\zeta}\cdot\eta)|0\rangle \tag{1.21}$$

and the action of the coordinate and momentum operators on this Dirac basis is

$$\hat{\eta}^a|\eta\rangle = \eta^a|\eta\rangle \qquad \hat{\zeta}_a|\eta\rangle = \frac{\partial_r}{\partial\eta^a}|\eta\rangle \tag{1.22}$$

where the subscript r (l) denotes the right (left) derivative.

However, now the construction of the matrix realization of the preceding paragraph makes no sense, unless we attach a degree to the basis vectors. To see this recall the fact that there exists a state of 'lowest weight' $|0\rangle$, which is annihilated by the coordinate operators, and from which the complete set of basis vectors $|n_1,\ldots,n_m\rangle$ can be generated according to (1.16) by repeated application of the momentum operators. Hence, if we attach the degree zero to the state $|0\rangle$, then the assignment of the degree $\sum_a n_a$ to $|n_1,\ldots,n_m\rangle$ makes the Dirac ket basis vector $|\eta\rangle$ an even quantity. Expressed in terms of the now Grassmann valued vector space basis, its expansion takes the form

$$|\eta\rangle = \sum_{n_1\cdots n_m}|n_1,\ldots,n_m\rangle(\eta^m)^{n_m}\cdots(\eta^1)^{n_1} \tag{1.23}$$

where here and below the ordering of the factors is essential.

The subtle point is to construct the corresponding bra vector $\langle\eta|$, which we want to yield the Grassmann δ-function

$$\langle\eta|\eta'\rangle = \delta(\eta-\eta') = (-1)^m(\eta-\eta')^1\cdots(\eta-\eta')^m \tag{1.24}$$

in analogy with the bosonic case; this has the naïve degree m so that the bra $\langle\eta|$ must have the same naïve degree. It is for this reason, that we cannot choose the conventional adjoint of $|\eta\rangle$; instead, we must define

$$\langle\eta| = \langle\bar{0}|(\hat{\eta}^1\cdots\hat{\eta}^m)\exp(\eta\cdot\hat{\zeta}). \tag{1.25}$$

The point of crucial importance with this definition is that we must give the dual $\langle\bar{0}|$ of $|0\rangle$ the degree zero in order to make sense, whereas the conventional counting for $|\bar{0}\rangle$ according to (1.18) yields m. Hence, we must alter the assignment of a degree to the adjoint basis since otherwise the quantity $\langle\bar{0}|0\rangle$ would be Grassmann valued.

This can consistently be done as follows. For this purpose, let us introduce the operator

$$\hat{G} = \sum_a \hat{\zeta}_a\hat{\eta}^a \tag{1.26}$$

which counts what we call the *Grassmann degree*. Its action on the basis (see (1.16)) is

$$\hat{G}|n_1,\ldots,n_m\rangle = \left(\sum_a n_a\right)|n_1,\ldots,n_m\rangle. \tag{1.27}$$

which is the same as the conventional naïve degree. Hence, for states we distinguish between the ghost number, being counted by $\hat{N}$, and the ghost degree, being counted by $\hat{G}$; its adjoint is $\hat{G}^* = m - \hat{G}$. Consequently, for its action on the adjoint basis $\langle 0|(\hat{\zeta}_m)^{n_m}\cdots(\hat{\zeta}_1)^{n_1}$, the conventional degree of which is $\sum_a n_a$, we obtain instead

$$\langle n_m,\ldots,n_1|\hat{G} = \langle n_m,\ldots,n_1|\left(\sum_a \bar{n}_a\right).$$

Furthermore, we pass to the dual basis (see (1.17) and (1.20))

$$\begin{aligned}\langle\overline{n_m,\ldots,n_1}| &= (-1)^{\sum_a \bar{n}_a(a-1)}\langle\bar{n}_m,\ldots,\bar{n}_1| \\ &= \langle\bar{0}|(\hat{\eta}^m)^{n_m}\cdots(\hat{\eta}^1)^{n_1}\end{aligned} \tag{1.28}$$

with the properties

$$\langle\overline{n_m,\ldots,n_1}|n'_1,\ldots,n'_m\rangle = \delta_{n_1nn'_1}\cdots\delta_{n_mn'_m} \tag{1.29}$$

$$\sum_{n_1\cdots n_m}|n_1,\ldots,n_m\rangle\langle\overline{n_m,\ldots,n_1}| = 1_{2^m}. \tag{1.30}$$

This dual basis then has the Grassmann degree

$$\langle\overline{n_m,\ldots,n_1}|\hat{G} = \langle\overline{n_m,\ldots,n_1}|\left(\sum_a n_a\right). \tag{1.31}$$

With this assignment, both $|0\rangle$ and $\langle\bar{0}|$ have degree zero, and thus $\langle\eta|$ has Grassmann degree m, as we wanted to achieve. Had we assigned to $\langle\bar{0}|$ the conventional degree m, then $\langle\bar{0}|0\rangle$ would also be of degree m, and we could not give this quantity the numerical value one.

It is straightforward now to show that the coordinate and momentum operators act on the Dirac bra vectors as

$$\langle\eta|\hat{\eta}^j = \langle\eta|\eta^j \qquad \langle\eta|\hat{\zeta}_i = -\frac{\partial_r}{\partial\eta^i}\langle\eta|. \tag{1.32}$$

Furthermore, one can prove the conjectured normalization property (1.24) of the Dirac basis by means of the explicit form

$$\begin{aligned}\langle\eta| &= \langle\bar{0}|(\hat{\eta}^1-\eta^1)\cdots(\hat{\eta}^m-\eta^m)\\ &= (-1)^{\binom{m}{2}}\sum_{n_1\cdots n_m}(-1)^{\sum_a \bar{n}_a(m-a+1)}(\eta^1)^{\bar{n}_1}\\ &\quad\cdots(\eta^m)^{\bar{n}_m}\langle\overline{n_m,\dots,n_1}|\end{aligned} \tag{1.33}$$

in its first version. The second version is needed to reduce the proof of the completeness relation for the Dirac basis, which is

$$(-1)^m\int d^m\eta|\eta\rangle\langle\eta| = 1 \tag{1.34}$$

to the completeness relation (1.20) of the Fock basis.

Let us relate these results to the Schrödinger wave function approach of the last but one subsection. For $\psi(\eta)$ (see (1.6)) with

$$|\psi\rangle = \int d^m\eta\,|\eta\rangle\psi(\eta) \tag{1.35}$$

we obtain from the completeness relation

$$\psi(\eta) = (-1)^m\langle\eta|\psi\rangle \tag{1.36}$$

and for the coefficients $\psi_{n_1\dots n_m}$, this gives

$$\psi_{n_1\dots n_m} = (-1)^{\binom{m+1}{2}}(-1)^{\sum_a n_a(m-a+1)}\psi(\bar{n}_1,\dots,\bar{n}_m) \tag{1.37}$$

which now come equipped with a Grassmann degree.

We end the discussion of the Dirac basis over configuration space with an investigation of the trace of an operator $\hat{O}$ with zero ghost number; this we define by means of the dual basis to be

$$\mathrm{Tr}\,\hat{O} = \sum_{n_1\cdots n_m}\langle\overline{n_m,\dots,n_1}|\hat{O}|n_1,\dots,n_m\rangle. \tag{1.38}$$

One can also introduce a supertrace, defined by

$$\mathrm{Str}\,\hat{O} = \sum_{n_1\cdots n_m}(-1)^{\sum_a n_a}\langle\overline{n_m,\dots,n_1}|\hat{O}|n_1,\dots,n_m\rangle. \tag{1.39}$$

By means of the Dirac basis, these traces can be expressed in the form

$$\mathrm{Tr}\,\hat{O} = \int d^m\eta\,\langle-\eta|\hat{O}|\eta\rangle \qquad \mathrm{Str}\,\hat{O} = \int d^m\eta\,\langle\eta|\hat{O}|\eta\rangle \tag{1.40}$$

as follows by a straightforward computation.

One can as well construct a Dirac basis in momentum space and introduce Fourier transformation; the relevant formulae are collected in an appendix.

1.4 Feynman type path integral

Having available the Dirac basis, the path integral treatment of the time-evolution operator for these unphysical fermions is rather straightforward; it closely follows the analogous bosonic case [15], and so we may be brief.

We assume the Hamiltonian $\hat{H} = H(\hat{\zeta},\hat{\eta})$ to be an even operator, the ordering being prescribed such that the momentum operators are placed to the left of the coordinate operators. The transition amplitude

$$\langle\eta''|\exp -i\hat{H}(t''-t')|\eta'\rangle = \langle t'',\eta''|\eta',t'\rangle \tag{1.41}$$

can be written in the form of a path integral as follows

$$\begin{aligned}\langle t'',\eta''|\eta',t'\rangle &= \lim_{\varepsilon\to 0}\int d\zeta_{N+1}\cdot d\zeta_N d\eta_N\cdots d\zeta_1 d\eta_1\\ &\quad\times\exp i\sum_{n=0}^{N}\Big(i\zeta_{n+1}\cdot(\eta_{n+1}-\eta_n)\\ &\quad-\varepsilon H(\zeta_{n+1},\eta_n)\Big)\end{aligned} \tag{1.42}$$

where $\eta_0 = \eta'$ and $\eta_{N+1} = \eta''$; note that there is an excess of one momentum integration. In formal continuum notation, this reads as

$$\langle t'',\eta''|\eta',t'\rangle = \int_{\eta'}^{\eta''} D[\zeta,\eta]\exp i\int_{t'}^{t''} dt\,(i\zeta\cdot\dot{\eta} - H(\zeta,\eta))\,. \tag{1.43}$$

As to be expected, the result looks rather similar to the bosonic Feynman path integral in hamiltonian form; we stress that only the discrete version, with the limit $\varepsilon\to 0$ taken afterwards, is well defined.

As an application, the transition amplitude can be computed exactly for a selfadjoint Hamiltonian of the form

$$\hat{H}(t) = \sqrt{-1}\hat{\zeta}_a\omega^a{}_b(t)\hat{\eta}^b \tag{1.44}$$

with $\omega(t)$ a real square m-matrix, which is assumed to be symmetric and may depend explicitly on time. Performing in the discrete version the integrations over (ζ_n,η_n) successively for $n = 1,\dots,N$, one ends up with

$$\begin{aligned}&\langle\eta''|P\exp -i\int_{t'}^{t''} dt\hat{H}(t)|\eta'\rangle\\ &\quad= \lim_{\varepsilon\to 0}\int d\zeta_{N+1}\exp\Big(-\zeta_{N+1}\cdot\eta_{N+1}\\ &\qquad+\zeta_{N+1}\cdot e^{\varepsilon\omega_N}\cdots e^{\varepsilon\omega_0}\eta_0\Big)\\ &\quad= \int d\zeta''\,\exp\Big(-\zeta''\cdot\eta''+\zeta''\cdot Pe^{\int_{t'}^{t''}\omega(t)dt}\eta'\Big)\end{aligned} \tag{1.45}$$

where $\omega_n = \omega(t_n)$ and P signifies the time ordering; the remaining integration over $\zeta'' = \zeta_{N+1}$ can also be done, and we obtain

$$\langle t'',\eta''|\eta',t'\rangle = \delta(\eta'' - Pe^{\int_{t'}^{t''}\omega(t)dt}\eta'). \tag{1.46}$$

Hence, only the 'classical' solution contributes to the transition amplitude. Finally, we can compute, e.g., the supertrace and obtain

$$\mathrm{Str}\, Pe^{-i\int_{t'}^{t''}\hat{H}(t)dt} = \left|1 - Pe^{\int_{t'}^{t''}\omega(t)dt}\right| \tag{1.47}$$

where here and below $|A|$ denotes the determinant of a square matrix A.

2 Path integral quantization of constrained systems

The path integral for ghost fermions is heuristically made use of in the BVF quantization of systems with first order constraints [2,19], without specifying its properties. We will show that it is the supertrace, which is used in this context, and what the reasons are why this must be so. Furthermore, we comment on the proof of the Fradkin-Vilkovisky theorem [3], which can be improved so as to stand objections.

So let a finite-dimensional Hamiltonian system be given, being subject to m bosonic first class constraints $\varphi_a(p,q)$; these are chosen to be momentum maps of an m-dimensional Lie algebra [1] so that $\{\varphi_a, \varphi_b\} = C^c{}_{ab}\varphi_c$ holds where the $C^c{}_{ab}$ are the structure constants; the Hamiltonian $H(p,q)$ is assumed to commute with the constraints. Furthermore, the auxiliary constraints χ^a are chosen to be holonomic since the constraints are cotangent lifts that are linear homogenous in the momenta; consistency then dictates the determinant $|\{\chi,\varphi\}|$ to be nonvanishing. An element of the extended supersymmetric phase space is denoted by $\xi = (p, q, \mu, \lambda, i\zeta, \eta, i\zeta^\times, \eta^\times)$, where the λ^a denote the Lagrange multipliers with corresponding momenta μ_a; the η^a and ζ_a the ghost coordinates and momenta, and analogously $\eta^{\times a}$ and $\zeta^\times{}_b$ for the antighosts. According to Fradkin and Vilkovisky [2], the path integral for this system is

$$Z_\chi = \int_{\mathrm{PBC}} d[p,q]d[\mu,\lambda]d[\zeta,\eta]d[\zeta^\times,\eta^\times]\exp iS_\chi \tag{2.1}$$

where the extended action takes the form

$$\begin{aligned}S_\chi = \int_{t'}^{t''} dt\Big(&p_i\dot{q}^i + \mu_a\dot{\lambda}^a + i\zeta_a\dot{\eta}^a + i\zeta^\times{}_a\dot{\eta}^{\times a}\\ &- H - i\{\Omega,\phi\}\Big).\end{aligned} \tag{2.2}$$

The meaning of the subscript PBC on the functional integration will be explained later; also the reasons for the rather special notation will become apparent below[1]. What remains to be specified is the BRST generator Ω and the gauge-fixing fermion ϕ. Here we depart from the standard choice since we take Ω to be

$$\Omega = \varphi_a(p,q)\eta^a + \frac{i}{2}\zeta_a C^a{}_{bc}\eta^b\eta^c + \mu_a\eta^{\times a} \tag{2.3}$$

whereas in the literature the nonminimal term is $\zeta^{\times a}\mu_a$; we shall comment on this discrepancy in a moment. Consequently, our gauge-fixing fermion differs as well from the standard choice:

$$\phi = \zeta_a\lambda^a + \zeta^\times{}_a\left(\chi^a - \frac{\xi}{2}\mu^a\right). \tag{2.4}$$

Let us first note that, on integrating out the antighost coordinates $\eta^\times$ and the ghost momenta ζ, the conventional form of the partition function in the derivative gauge is obtained; in particular, for the Yang-Mills case, the Faddeev-Popov path integral [31,32] is regained. Hence, our modification does not alter the final results.

The essential differences come in if the basic properties of the path integral are investigated. The main point in the original work of Fradkin and Vilkovisky [2] is the statement that the functional integral for a system with first class constraints, as shown in (2.1) and (2.2), does not depend on the special choice of the gauge-fixing fermion. A proof of this statement, known as the Fradkin-Vilkovisky theorem, was later given by Batalin and Vilkovisky [3], and since then it has often been repeated.

The actual proof, however, requires modification for two reasons. The first is that the notorious choice made for the nonminimal contribution to the BRST charge in the literature is the term $\zeta^{\times a}\mu_a$, which is quadratic in the momenta; it is to be contrasted with our choice (2.3), being linear in the momenta. This is a crucial point, since a term quadratic in the momenta prevents the functional integral from being invariant under BRST transformations. The reason is that nonvanishing boundary terms get involved, which destroy its invariance. These boundary terms were discussed by Henneaux and Teitelboim ([11,19,21], see also [5]) in the attempt to invent boundary conditions which enforce the vanishing of the terms in question. For our choice of the nonminimal term, such boundary terms will be absent. Second, for the transformation introduced by Batalin and Vilkovisky in the attempt to demonstrate the independence of the choice of the gauge fermion there is no reason to believe the extended action to be invariant since the corresponding generating function yields a transformation, which is nonlocal. Below we give a proof of the Fradkin-Vilkovisky theorem which does not suffer from these defects.

Let us first comment on the boundary term; for this purpose, the following simplifying notation is introduced (see, e.g., [33]). We collectively denote coordinates of superconfiguration space by $z = (q, \lambda, \eta, \eta^\times)$ and the corresponding momenta by $\pi = (p, \mu, i\zeta, i\zeta^\times)$ so that the BFV partition function takes the form

$$Z_\phi(t_2,t_1) = \int d[\pi,z]\,\exp i\int_{t_1}^{t_2} dt(\pi\dot{z} - H_\phi(\pi,z)) \tag{2.5}$$

[1] The relation to the notation of Henneaux and Teitelboim [19] is $G_a \equiv \varphi_a$ for the constraints, $\lambda_a \equiv \lambda_a$ and $b_a \equiv \mu_a$ for the multipliers and their conjugate momenta, $\eta_a \equiv \eta_a$ and $\mathcal{P}_a \equiv \zeta_a$ for the ghosts, and $\overline{C}_a \equiv \eta^\times{}_a$ and $\rho_a \equiv \zeta^\times{}_a$ for the antighosts

with

$$H_\phi = H + i\{\Omega, \phi\}. \qquad (2.6)$$

Here the BRST charge $\Omega = \Omega^*$ is an odd phase space function with $\{\Omega, \Omega\} = 0$ which commutes with the Hamiltonian H in the sense $\{\Omega, H\} = 0$, and as such is a conserved quantity; it contains the original first class constraints φ_a, whereas the gauge-fixing fermion ϕ depends on the gauge conditions χ^a. Under a supercanonical transformation with (even) infinitesimal generating function δG, the coordinates and momenta transform according to

$$z^A \mapsto z^A + \frac{\partial_l \delta G}{\partial \pi_A} \qquad \pi_A \mapsto \pi_A - \frac{\partial_r \delta G}{\partial z^A} \qquad (2.7)$$

and the partition function is transformed into

$$Z_\phi(t_2, t_1) = \int d[\pi, z] \exp i \int_{t_1}^{t_2} dt \Bigg(\pi \dot{z} - H_\phi(\pi, z) + \frac{d}{dt}\left(\pi_A \frac{\partial_l \delta G}{\partial \pi_A} - \delta G\right) - \{\delta G, H_\phi\} \Bigg) \qquad (2.8)$$

where we have used that the super–Liouville measure $d(\pi, z)$ is invariant against general supercanonical BRST transformations. Hence, for the functional integral to be invariant, we must also guarantee that the extended Hamiltonian is invariant, i.e. $\{\delta G, H_\phi\} = 0$, and that the boundary term vanishes

$$\left(\pi_A \frac{\partial_l \delta G}{\partial \pi_A} - \delta G\right)\Bigg|_{t_1}^{t_2} = 0. \qquad (2.9)$$

In the present case, the supercanonical transformations have the special form $\delta G = \delta\theta\Omega$ with Ω the nilpotent BRST generator and $\delta\theta$ a purely imaginary Grassmann parameter. Then the first requirement is met by construction. As to the boundary term, this vanishes if δG is linear homogeneous in the supermomenta; correspondingly, boundary terms are absent. It is this reason where our request for the BRST generator to be linear in the supermomenta comes from.

Actually, one could weaken the above requirement. A linear, but inhomogeneous generating function of the form $G = X^A(z)\pi_A + \Lambda(z)$ would also do since $z^A(t_1) = z^A(t_1)$ on account of the fact that the functional integral is the trace of the time evolution operator in the bosonic and the supertrace in the fermionic sector. In particular, for a Yang-Mills theory, such an inhomogeneous term is absent; if it were present, one would face the possibility of an anomalous symmetry.

For the proof of the Fradkin-Vilkovisky theorem, it will turn out to be advantageous to use symplectic notation. An element of the supersymmetric phase space is denoted by $\xi = (\pi, z)$ and the graded Poisson bracket is

$$\{f, g\} = \frac{\partial_r f}{\partial \xi^\alpha} \omega^{\alpha\beta} \frac{\partial_l g}{\partial \xi^\beta} \qquad (2.10)$$

where the (co-)symplectic supermatrix with entries $\omega^{\alpha\beta}$ is antisupersymmetric. Consider then the transformation $\xi'^\alpha = \xi^\alpha + \delta\theta(\xi)\{\Omega(\xi), \xi^\alpha\}$ where the Grassmann parameter $\delta\theta$ is now taken to be ξ-dependent. Accordingly, this is not a canonical transformation, as can be seen by computing the superdeterminant of the superjacobian, which is

$$\text{Sdet}\left(\frac{\partial_r \xi'}{\partial \xi}\right) = 1 - \{\delta\theta, \Omega\} = \exp -\{\Omega, \delta\theta\}. \qquad (2.11)$$

We now choose

$$\delta\theta(\xi) = i\delta\phi(\xi)\Delta t \qquad (2.12)$$

which indeed is a purely imaginary quantity, and so we find for the functional measure in its defining discrete version

$$\prod_{n=0}^{N} \text{Sdet}\left(\frac{\partial_r \xi'}{\partial \xi}\right)_n = \exp -i \sum_n \Delta t_n \{\Omega, \delta\phi\}(\xi_n) \equiv \exp -i \int_{t_1}^{t_2} dt \{\Omega, \delta\phi\} \quad : \Delta t \to 0 \qquad (2.13)$$

Let us comment at this point on the notorious choice made in the literature, which instead of (2.12) is

$$\delta\theta[\xi] = i \int_{t'}^{t''} \delta\phi(\xi(t)) dt.$$

With this construct, however, δG is a nonlocal generating function that makes no sense in the Hamiltonian formalism. What remains finally to be shown is that the transformation

$$\xi_n'^\alpha = \xi_n^\alpha - i\delta\phi(\xi_n)\{\Omega(\xi_n), \xi_n^\alpha\}\Delta t_n$$

leaves the discrete action

$$\sum_{n=0}^{N} \Big(\pi_n(z_{n+1} - z_n) - \Delta t_n H(\pi_n, z_{n-1}) - i\Delta t_n \{\Omega, \phi\}(\pi_n, z_{n-1}) \Big)$$

invariant; but there is nothing to prove since $\delta\pi_n$ and δz_n are proportional to Δt_n, and so yield no contribution in the continuum limit. Hence, again there are no boundary terms to be discussed away. We thus have proven that $Z_\phi(t_2, t_1) = Z_{\phi+\delta\phi}(t_2, t_1)$ and since finite changes are obtained by exponentiation, the partition function is independent of the choice of the gauge-fixing fermion. But this statement requires qualification; it only holds for homotopic gauge fermions.

3 Operator approach to BVF systems

Up to now we have not commented on the paths that enter the BVF path integral. In order to discuss this question, we need the operator treatment to BRST quantization.

Hence, the quantity of interest is the time-evolution operator

$$\hat{U}_\chi(t) = \exp -i(\hat{H} + [\hat{\Omega}, \hat{\phi}])t \tag{3.1}$$

which is BRST invariant. We assume the Hamiltonian $\hat{H}$ to be Weyl ordered so that the midpoint rule is the correct prescription in the path integral. The gauge-fixing fermion presents no ordering ambiguities, but the BRST operator does. These are also overcome by means of the Weyl ordering, which results in the symmetric ordering for the constraints $\varphi_a(p, q)$ since these are linear homogeneous in the momenta by assumption; then the algebraic properties of the constraints remain unaltered at the operator level. Furthermore, if we define the ghost number operator to be

$$\hat{N} = \frac{1}{2}(\hat{\eta}^a \hat{\zeta}_a - \hat{\zeta}_a \hat{\eta}^a) + \frac{1}{2}(\hat{\eta}^{\times a} \hat{\zeta}^\times{}_a - \hat{\zeta}^\times{}_a \hat{\eta}^{\times a}) \tag{3.2}$$

then $\hat{\Omega}$ has ghost number $+1$ and $\hat{\phi}$ ghost number -1 so that the time-evolution operator has ghost number zero.

For such operators, we know from Schwarz's work [22] that the supertrace over the extended state space reduces to the supertrace over the cohomology groups. In explicit terms, let $\hat{O}$ be a BRST invariant operator of zero ghost number. Since the number operator defines a grading of the (extended) state space $V = \oplus_l V_l$ with $-m/2 \leq l \leq +m/2$, we can define the restriction $\hat{\Omega}_l$ of the BRST operator to the subspace V_l; then the lth cohomology group is defined as the quotient $H^l(\Omega) = \mathrm{Ker}(\hat{\Omega}_l)/\mathrm{Im}(\hat{\Omega}_{l-1})$. This is the subspace of V_l, consisting of physical states $\Omega\psi = 0$ modulo exact states. The Lefschetz formula [22] then says that

$$\mathrm{Str}\hat{O} = \bigoplus_l (-1)^l \mathrm{Tr}_{V_l} \hat{O} = \bigoplus_l (-1)^l \mathrm{Tr}_{H^l(\Omega)} \hat{O} \tag{3.3}$$

since the contributions from non-closed states cancel against those from exact states.

Hence, in the ghost fermion sector we must choose the supertrace of the time-evolution operator

$$Z_\chi(t) = \mathrm{Tr}_{\mathrm{B}} \mathrm{Str}_{\mathrm{GF}} \exp -i(\hat{H} + [\hat{\Omega}, \hat{\phi}])t \tag{3.4}$$

since we want only those states to contribute that obey the generalized Dirac constraint $\hat{\Omega}\psi(q, \lambda, \eta, \eta^\times) = 0$. It is of crucial importance to note that the traces are taken over the full extended state space, since the restriction to the cohomological subspaces is automatic, so that no normalization problems are encountered for the partition function. Furthermore, as we have shown in the last but one section (see (1.40)), it is the supertrace for the ghost fermions that translates into periodic boundary conditions in the functional integral, and this is the meaning of the subscript PBC in (2.1). Also note that the boundary values $\eta(t'') = \eta(t')$ and $\eta^\times(t'') = \eta^\times(t')$ are integrated over and thus cannot be chosen to be zero, as is done elsewhere.

In addition to the physical state condition, a further restriction is needed in order to eliminate negative norm states. This is usually achieved by requiring physical states to have ghost number zero. But often this requirement is too restrictive; e.g., for the open bosonic string (see [34]) the relevant cohomologies are at the values $\pm 1/2$. In general, the correct choice of the relevant cohomology group depends on the system under consideration (see also the concluding remark in [35]); there is no model independent proof of a no-ghost theorem.

We return to the BFV system of the form considered in the preceding section. In this case, the relevant cohomology group is indeed given by $H^0(Q)$ since the total number of constraints is $2m$, i.e., an even number. Nevertheless, the admissible states have zero norm. This is seen on determining those wave functions of ghost number zero that obey $\hat{\Omega}\psi(q, \lambda, \eta, \eta^\times) = 0$; they are obtained to be

$$\begin{aligned} \psi_{0;m}(q, \lambda, \eta, \eta^\times) &= \psi_{0;m}(q)\eta^{\times 1} \cdots \eta^{\times m} \\ \psi_{m;0}(q, \lambda, \eta, \eta^\times) &= \psi_{0;m}(q)\eta^m \cdots \eta^1 \end{aligned} \tag{3.5}$$

with $\hat{\varphi}_a \psi_{0;m}(q) = 0$ and $\hat{\varphi}_a \psi_{0;m}(q) = 0$. As follows from the Berezin rules, these wave functions indeed have vanishing norm. In such a situation it is usually argued in the literature that the norm is of the form $0 \cdot \infty$ since the Grassmann integration gives 0 and the integration over q yields ∞ because one integrates over wave functions obeying $\hat{\varphi}_a \psi(q) = 0$, and this results in an ill-defined expression. However, in view of what we have shown, this contradiction is void since the (super) trace is taken over the full extended state space $\mathcal{H}_{\mathrm{ext}}$; in addition, the trace is constructed by means of $\mathcal{H}_{\mathrm{ext}}$ and its dual $\mathcal{H}^*_{\mathrm{ext}}$ so that the (indefinite) inner product does not get involved at all. This is a fact of crucial importance, which is also confirmed by going through the proof of the Lefschetz formula. Hence, we have a probabilistic interpretation because the states $\psi_{0;m}$ and $\psi_{m;0}$ are then paired in duality (if the relevant cohomology appears at a nonzero value l one must invoke the duality $H^{+l}(\Omega) \cong H^{-l}(\Omega)$) so that

$$\begin{aligned} \langle \psi_{m;0} | \psi'_{0;m} \rangle &= \int dq d\eta d\eta^\times (\psi(q)\eta^m \cdots \eta^1)^* \\ &\quad \times \left(\psi'(q)\eta^{\times 1} \cdots \eta^{\times m}\right) \\ &= \int dq \psi^*(q) \psi'(q) \end{aligned} \tag{3.6}$$

where $\psi_{m;0}(q) = \psi(q) = \psi_{0;m}(q)$ (cf. also [36]). Since the cancellation of the unphysical states in the (super) trace is automatic, there are also no normalization problems for the functional integral. This is the version of the no ghost theorem for a system with first class constraints.

What remains to resolve is the intriguing problem that the functional integral simultaneously takes care of all cohomology groups and not, as one would wish, of the zero cohomology only. One might argue that the operator $[\hat{\Omega}, \hat{\phi}]$, or its exponentiated version entering the time-evolution operator, will be of special relevance. This operator has been introduced by Batalin and Marnelius ([7], see also [6]) in the attempt to construct an inner product for constraints having a continuous spectrum. For the case at hand, it has recently been discussed by Rogers [8]. In this latter work, the proposal is made that the operator

$[\hat{\Omega}, \hat{\phi}]$ could provide for the mechanism that only the zero cohomology survives in the partition function. This conjecture rests on the remarkable fact that, if the operator in question were invertible on all physical states, then there would be no cohomology at all; the proof is straightforward and amounts to showing that, under the above hypothesis, all physical states are also exact. The legitimate conclusion, as drawn by Rogers, is that the gauge-fixing fermion should be chosen such that the only states on which $[\hat{\Omega}, \hat{\phi}]$ is not invertible are the elements of $H^0(\Omega)$. But in our case there is (up to the choice of χ) no freedom in disposing of $\hat{\phi}$; beyond that, on the states $\psi_{m;0}$ and $\psi_{0;m}$ the action of $[\hat{\Omega}, \hat{\phi}]$ is definitely nonzero since the operator $[\hat{\chi}, \hat{\varphi}]$ is invertible; of course, the latter property only holds modulo Gribov obstructions [37,38] (see also [39]). Hence, for the case at hand (and, in particular, for Yang-Mills theory) this kind of approach does not work.

As we see it, there is no hope that the functional integral manages by itself that only the zero cohomology survives. Hence, it is forced upon us to introduce a thermodynamic potential $\gamma \in \mathbb{R}$ for the ghosts, and so we must consider the generalization

$$Z_\chi(t, \gamma) = \mathrm{Tr}_{\mathrm{B}}\mathrm{Str}_{\mathrm{GF}} \exp -i(\hat{H} + [\hat{\Omega}, \hat{\phi}] + i\gamma \hat{N})t \quad (3.7)$$

which makes sense since the ghost number operator commutes with both $\hat{H}$ and $[\hat{\Omega}, \hat{\phi}]$. Again, the partition function may be written as a path integral

$$\begin{aligned} Z_\chi(t'' - t', \gamma) = \int_{\mathrm{PBC}} & d[p,q] d[\mu, \lambda] d[\zeta, \eta] d[\zeta^\times, \eta^\times] \\ & \times \exp i \int_{t'}^{t''} dt \Big(p\dot{q} + \mu\dot{\lambda} + i\zeta\dot{\eta} \\ & + i\zeta\dot{\eta} - H - i\{\Omega, \phi\} - i\gamma N \Big) \end{aligned} \quad (3.8)$$

where the restriction to ghost number zero is obtained by expanding in terms of γ and retaining the γ-independent term only. Hence, in the end, the thermodynamic potential for the ghosts is no longer visible; but it should tacitly be assumed as present.

4 Concluding remarks

Finally, we want to comment on the subtleties that may arise in solving the Dirac condition for physical states, after the ghost degrees of freedom have been eliminated. As we shall see, topological properties in the large then get involved, which may give rise to anomalies of the quantum system.

We do this on the example of pure abelian Chern-Simons theory (see, e.g., [40]) with the action

$$S = \frac{k}{4\pi} \int dt \int_\Sigma d^2x\, \varepsilon^{ij} (\dot{A}_i A_j + A_0 F_{ij}) \quad (4.1)$$

where the two-dimensional domain Σ is chosen to be either the whole plane or the torus. This is an action in Hamiltonian first order form so that the kinetic term determines the symplectic two-form, from which the canonical Poisson brackets can be read off to be (see also [41, 42])

$$\{A_i(x), A_j(y)\} = -\frac{2\pi}{k} \varepsilon_{ij} \delta(x-y). \quad (4.2)$$

Since we want to quantize the system, a polarization [1] must be chosen. There is no natural real polarization available, but the Levi-Civita tensor ε^{ij} gives rise to a complex structure. Hence, we choose holomorphic quantization with[2]

$$\hat{A}_{\bar{z}} = A_{\bar{z}} \qquad \hat{A}_z = \frac{\pi}{k} \frac{\delta}{\delta A_{\bar{z}}} \quad (4.3)$$

and the Bargmann inner product for Schrödinger wave functionals $\psi[A_{\bar{z}}]$ then is

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle = \int & d[A_{\bar{z}}, A_z] \\ & \times \exp\left(-\frac{k}{\pi} \int d^2x A_{\bar{z}} A_z \right) \overline{\psi_1[A_{\bar{z}}]}\, \psi_2[A_{\bar{z}}]. \end{aligned} \quad (4.4)$$

What the second term in (4.1) tells us is that we have a system with the first class constraint

$$\hat{C} = i\frac{k}{2\pi} \hat{F}_{12} = i\frac{k}{\pi} (\partial_{\bar{z}} \hat{A}_z - \partial_z \hat{A}_{\bar{z}}) \quad (4.5)$$

where the time component A_0 of the gauge field serves as a Lagrange multiplier. Since, classically, the constraint $F_{12} = 0$ only leaves gauge degrees of freedom, the system appears to be trivial, but quantum mechanically it is not if we quantize first and constrain afterwards. Furthermore, the Hamiltonian is identically zero; hence, there is also no evolution in time so that we face a purely cohomological problem.

Physical wave functionals must obey the Dirac condition $\hat{C}(x)\psi[A_{\bar{z}}] = 0$; but instead of trying to solve this by direct attack, we make a digression and look at the constraint as a symmetry transformation. By exponentiation, we obtain the operator

$$\hat{U}[g] = \exp\left(-i \int d^2x\, \alpha\, \hat{C} \right) \quad (4.6)$$

with $g = \exp(-i\alpha) \in U(1)$, the action of which on Schrödinger wave functionals is calculated to be

$$\hat{U}[g]\psi[A_{\bar{z}}] = \exp(-iW[A_{\bar{z}}; g])\, \psi[g^{-1}A_{\bar{z}}] \quad (4.7)$$

where $gA_{\bar{z}} = A_{\bar{z}} + \partial_{\bar{z}}\alpha$. Were it not for the exponential prefactor, this would be the standard behavior of the wave functional under time-independent gauge transformations. Instead, we have a projective transformation law

[2] The conventions are $z = x^1 + ix^2$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$

with the Lie group 1-cochain (as opposed to the Lie algebra cochains having been encountered in the preceding sections)

$$W[A_{\bar{z}}, g] = \frac{k}{\pi}\int d^2x A_{\bar{z}}\partial_z\alpha - \frac{k}{2\pi}\int d^2x\partial_{\bar{z}}\alpha\partial_z\alpha \quad (4.8)$$

which, potentially, is anomalous. We postpone the discussion of its properties and return to (4.7); the integrated version of the Dirac condition then is

$$\hat{U}[g]\psi[A_{\bar{z}}] = \psi[A_{\bar{z}}]. \quad (4.9)$$

In the plane, this can be solved on passing to the complexification of $U(1)$; the result is

$$\psi[A_{\bar{z}}] = \exp iW[A_{\bar{z}}] \quad (4.10)$$

with

$$W[A_{\bar{z}}] = -i\frac{k}{2\pi}\int d^2x A_{\bar{z}}(x)\partial_z P(x, y)A_{\bar{w}}(y) \quad (4.11)$$

where $P(x-y) = -4\partial_z G(x-y)$ denotes the Green's function of the operator $\partial_{\bar{z}}$, and $G(x-y) = -\frac{1}{4\pi}\log|\mu(z-w)|^2$ the standard propagator with μ an infrared cutoff. Despite the fact that the wave function solves the Dirac condition, nevertheless, it has finite norm with respect to the Bargmann inner product; thus, no normalization problems (cf. the remarks in the preceding section) arise.

Hence, the theory is exactly solvable but, as the transformation law (4.7) exhibits, the behaviour of wave functionals under gauge transformations is nonstandard. Whether it is truly anomalous or not, this depends on the 1-cochain (for relevant background, see [43,44]). So we need compute the coboundary

$$\begin{aligned}(\Delta W)[A_{\bar{z}}; g, h] &= W[g^{-1}A_{\bar{z}}; h] + W[A_{\bar{z}}; g] - W[A_{\bar{z}}; gh] \\ &= \frac{1}{2}[\hat{Q}[\alpha], \hat{Q}[\beta]]\end{aligned} \quad (4.12)$$

where $\hat{Q}[\alpha] = \int d^2x\,\alpha\,\hat{C}$. The right-hand side is the commutator of two abelian generators, and one expect this to vanish; if so, one can look at (4.12) as the integrated form of a Wess-Zumino consistency condition [45].

In the plane, this is indeed correct. Moreover, the 1-cocycle is even exact since

$$W[A_{\bar{z}}; g] = W[g^{-1}A_{\bar{z}}] - W[A_{\bar{z}}]. \quad (4.13)$$

Hence, we can pass to $\psi'[A_{\bar{z}}] = \exp(-iW[A_{\bar{z}}])\ \psi[A_{\bar{z}}]$ with conventional transformation law $\hat{U}'[g]\psi'[A_{\bar{z}}] = \psi'[g^{-1}A_{\bar{z}}]$ since the 1-coboundary disappears.

On the torus, however, the boundary of the 1-cochain is nonzero so that the Wess-Zumino consistency condition no longer holds. The reason is that, despite naïve expectation, the commutator on the right side of (4.12) need not vanish. This happens for large gauge transformations

$$g_m(x) = \exp\left(-i2\pi\left(m_1\frac{x_1}{L_1} + m_2\frac{x_2}{L_2}\right)\right) \quad (4.14)$$

where $m = (m_1, m_2)$ with m_1 and m_2 integer, which are not continuously connected to the identity. Here, the torus is considered as a rectangle $L_1 \times L_2$ with opposite points identified; hence, boundary conditions become important. In particular, the correct choice of the generator of gauge transformations now is

$$\hat{Q}[\alpha] = \int d^2x\left(\partial_{\bar{z}}\alpha\,\hat{A}_z - \partial_z\alpha\,\hat{A}_{\bar{z}}\right) \quad (4.15)$$

which differs from the earlier form in decisive boundary terms; one then finds for the coboundary

$$(\Delta W)[A_{\bar{z}}; g_m, g_n] = 2\pi ik\, m \times n. \quad (4.16)$$

The coupling constant generally takes values $k = r/s$ with r and s coprime integers; in particular, the case $r = 1$ is of special relevance. This entails that, quantum mechanically, the product of two (classically commuting) large gauge transformations do not commute.

What this result shows is that pure abelian Chern-Simons in a finite geometry becomes truly anomalous since the abelian large gauge transformations, one begins with classically, no longer commute at the quantum level.

In concluding, let us mention this is not the end of the story. As has been shown elsewhere [46,47], the noncommutative behaviour of large gauge transformations leads to a quantum symmetry [48]. Hence, it appears that anomalies should also admit an interpretation in terms of quantum symmetries.

Appendix

In this appendix some formulae for Dirac states of ghost fermions over momentum space and Fourier transformation are collected (cf. also [49]). We begin with the definition

$$|\zeta\rangle = \exp(\hat{\eta}\cdot\zeta)(\hat{\zeta}_m\cdots\hat{\zeta}_1)|0\rangle \quad \langle\zeta| = \langle\bar{0}|\exp(\zeta\cdot\hat{\eta}) \quad (A.1)$$

where now the Dirac ket has Grassmann degree m, and the dual bra has Grassmann degree zero. On these states, the momentum and coordinate operators act as follows

$$\begin{aligned}\hat{\zeta}_a|\zeta\rangle &= \zeta_a|\zeta\rangle & \hat{\eta}^a|\zeta\rangle &= +\frac{\partial_r}{\partial\zeta_a}|\zeta\rangle \\ \langle\zeta|\hat{\zeta}_a &= \langle\zeta|\zeta_a & \langle\zeta|\hat{\eta}^a &= -\frac{\partial_r}{\partial\zeta_a}\langle\zeta|.\end{aligned} \quad (A.2)$$

The normalization is

$$\langle\zeta|\zeta'\rangle = (-1)^m\delta(\zeta-\zeta') = (\zeta_m-\zeta'_m)\cdots(\zeta_1-\zeta'_1) \quad (A.3)$$

and the completeness relation takes the form

$$(-1)^m\int d^m\zeta\,|\zeta\rangle\,\langle\zeta| = 1. \quad (A.4)$$

The overlap with the configuration space basis turns out to be

$$\langle\zeta|\eta\rangle = \exp(\eta\cdot\zeta) \qquad \langle\eta|\zeta\rangle = \exp(\zeta\cdot\eta) \quad (A.5)$$

and these bases are connected as follows

$$
\begin{aligned}
|\zeta\rangle &= (-1)^m \int d^m\eta \, \exp(-\zeta\cdot\eta)|\eta\rangle \\
|\eta\rangle &= (-1)^m \int d^m\zeta \, \exp(-\eta\cdot\zeta)|\zeta\rangle
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\langle\zeta| &= (-1)^m \int d^m\,\eta \exp(\zeta\cdot\eta)\langle\eta| \\
\langle\eta| &= \int d^m\,\zeta \exp(\eta\cdot\zeta)\langle\zeta| .
\end{aligned}
\tag{A.7}
$$

We define $\psi(\zeta)$ through

$$
|\psi\rangle = \int d^m\zeta \, |\zeta\rangle \, \psi(\zeta) \tag{A.8}
$$

and since $|\zeta\rangle$ is even, we have $\langle\zeta|\psi\rangle = (-1)^m\psi(\zeta)$ so that the Fourier transform and its inverse are given by

$$
\begin{aligned}
\psi(\zeta) &= (-1)^m \int d^m\eta \, \exp(\zeta\cdot\eta)\psi(\eta) \\
\psi(\eta) &= \int d^m\zeta \, \exp(\eta\cdot\zeta)\psi(\zeta)
\end{aligned}
\tag{A.9}
$$

with the conventions $d^m\zeta = d\zeta^1\cdots d\zeta^m$ and $d^m\eta = d\eta^m \cdots d\eta^1$; this is the definition of the Fourier transform that is used in the main text.

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